A NOTE ON THE FROBENIUS-EULER NUMBERS AND POLYNOMIALS ASSOCIATED WITH BERNSTEIN POLYNOMIALS

SERKAN ARACI AND **MEHMET ACIKGOZ**

Abstract. The present paper deals with Bernstein polynomials and Frobenius-Euler numbers and polynomials. We apply the method of generating function and fermionic p-adic integral representation on $\mathbb{Z}_p$, which are exploited to derive further classes of Bernstein polynomials and Frobenius-Euler numbers and polynomials. To be more precise we summarize our results as follows, we obtain some combinatorial relations between Frobenius-Euler numbers and polynomials. Furthermore, we derive an integral representation of Bernstein polynomials of degree n on $\mathbb{Z}_p$. Also we deduce a fermionic p-adic integral representation of product Bernstein polynomials of different degrees $n_1, n_2, \cdots$ on $\mathbb{Z}_p$ and show that it can be written with Frobenius-Euler numbers which yields a deeper insight into the effectiveness of this type of generalizations. Our applications possess a number of interesting properties which we state in this paper

1. **Introduction and Notations**

Let p be a fixed odd prime number. Throughout this paper we use the following notations. By $\mathbb{Z}_p$ we denote the ring of p-adic rational integers, $\mathbb{Q}$ denotes the field of rational numbers, $\mathbb{Q}_p$ denotes the field of p-adic rational numbers, and $\mathbb{C}_p$ denotes the completion of algebraic closure of $\mathbb{Q}_p$. Let $\mathbb{N}$ be the set of natural numbers and $\mathbb{N}^* = \mathbb{N} \cup \{0\}$. The p-adic absolute value is defined by

$$|p|_p = \frac{1}{p}.$$

In this paper, we assume $|q-1|_p < 1$ as an indeterminate. In [17-19], let $UD(\mathbb{Z}_p)$ be the space of uniformly differentiable functions on $\mathbb{Z}_p$. For $f \in UD(\mathbb{Z}_p)$, the fermionic p-adic integral on $\mathbb{Z}_p$ is defined by T. Kim:

$$I_{-1}(f) = \int_{\mathbb{Z}_p} f(\xi)\, d\mu_{-1}(\xi) = \lim_{N\to\infty} \sum_{\xi=0}^{p^N-1} f(\xi)(-1)^{\xi}. \tag{1.1}$$

From (1.1), we have well known the following equality:

$$I_{-1}(f_1) + I_{-1}(f) = 2f(0) \tag{1.2}$$

here $f_1(x) := f(x+1)$ (for details, see[3-24]).

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Let $C([0,1])$ be the space of continuous functions on $[0,1]$. For $C([0,1])$, the Bernstein operator for f is defined by

$$B_n(f,x)=\sum_{k=0}^{n}f\left(\frac{k}{n}\right)B_{k,n}(x)=\sum_{k=0}^{n}f\left(\frac{k}{n}\right)\binom{n}{k}x^k(1-x)^{n-k}$$

where $n,\,k\in\mathbb{Z}_+:=\{0,1,2,3,...\}$. Here $B_{k,n}(x)$ is called Bernstein polynomials, which are defined by

$$B_{k,n}(x)=\binom{n}{k}x^k(1-x)^{n-k},\ x\in[0,1] \tag{1.3}$$

(for more informations on this subject, see [1-6, 11, 14, 15, 17, 21-24])

In [7], as is well known, Frobenius-Euler polynomials are defined by means of the following generating function:

$$\sum_{n=0}^{\infty}H_n(u,x)\frac{t^n}{n!}=e^{H(u,x)t}=\frac{1-u}{e^t-u}e^{xt}. \tag{1.4}$$

where the usual convention about replacing $H^n(u,x)$ by $H_n(u,x)$. For $x=0$ in (1.4), we have to $H_n(u,0):=H_n(u)$, which is called Frobenius-Euler numbers. Then, we can write the following

$$e^{H(u)t}=\sum_{n=0}^{\infty}H_n(u)\frac{t^n}{n!}=\frac{1-u}{e^t-u}. \tag{1.5}$$

By (1.4) and (1.5), we easily see the following applications:

$$\begin{aligned} e^{(H(u)+1)t}-ue^{H(u)t} &= 1-u \\ \sum_{n=0}^{\infty}\left[(H(u)+1)^n-uH_n(u)\right]\frac{t^n}{n!} &= 1-u \end{aligned}$$

After these applications, we derive the following Lemma.

Lemma 1. *For $|u|>1$ and $n\in\mathbb{Z}_+:=\mathbb{N}\bigcup\{0\}$, we have*

$$(H(u)+1)^n-uH_n(u)=\begin{cases}1-u, & \text{if } n=0\\ 0, & \text{if } n\neq 0.\end{cases} \tag{1.6}$$

In this paper, we obtained some relations between the Frobenius-Euler numbers and polynomials and the Bernstein polynomials. From these relations, we derive some interesting identities on the Frobenius-Euler numbers.

2. On the Frobenius-Euler numbers and polynomials

Let us take $f(x)=u^xe^{tx}$ in (1.1), by (1.2), we see that

$$\int_{\mathbb{Z}_p}u^{\eta}e^{\eta t}d\mu_{-1}(\eta)=\frac{2}{1+u}H_n\left(-u^{-1}\right). \tag{2.1}$$

By (1.4) and (2.1), we have the following theorem.

Theorem 1.

$$\int_{\mathbb{Z}_p}u^{\eta}(x+\eta)^n d\mu_{-1}(\eta)=\frac{2}{u+1}H_n\left(-u^{-1},x\right). \tag{2.2}$$

By applying some combinatorial techniques in (2.2), we derive the following

$$\int_{\mathbb{Z}_p} u^{\eta} (x+\eta)^n \, d\mu_{-1}(\eta) = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} \left\{ \int_{\mathbb{Z}_p} u^{\eta} \eta^k d\mu_{-1}(\eta) \right\}.$$

So, from above, we have the well known identity

$$H_n\left(-u^{-1}, x\right) = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} H_k\left(-u^{-1}\right) = \left(H\left(-u^{-1}\right) + x\right)^n. \tag{2.3}$$

by using the *umbral*(symbolic) convention $H^n(u) := H_n(u)$.

The Frobenius-Euler polynomials have to symmetric properties, which is shown by Choi *et al*. in [7], as follows:

$$H_n\left(-u^{-1}, 1-x\right) = (-1)^n H_n\left(-u^{-1}, x\right).$$

For $n \in \mathbb{N}$, by (2.3), Choi *et al*. derived the following equality:

$$u^2 H_n\left(-u^{-1}, 2\right) = u^2 + u + H_n\left(-u^{-1}\right). \tag{2.4}$$

From (2.2) and (2.4), we easily see that

$$\begin{aligned} &\int_{\mathbb{Z}_p} u^{\eta} (1-\eta)^n \, d\mu_{-1}(\eta) \\ = \; & (-1)^n \int_{\mathbb{Z}_p} u^{\eta} (\eta-1)^n \, d\mu_{-1}(\eta) \\ = \; & \frac{2}{u+1} (-1)^n H_n\left(-u^{-1}, -1\right) \\ = \; & \frac{2}{u+1} H_n\left(-u^{-1}, 2\right). \end{aligned} \tag{2.5}$$

Thus, we obtain the following Theorem.

Theorem 2. *The following identity*

$$\int_{\mathbb{Z}_p} u^{\eta} (1-\eta)^n \, d\mu_{-1}(\eta) = \frac{2}{u+1} H_n\left(-u^{-1}, 2\right) \tag{2.6}$$

is true.

Let $n \in \mathbb{N}$. By expression of (2.4) and (2.6), we get

$$\int_{\mathbb{Z}_p} u^{\eta} (1-\eta)^n \, d\mu_{-1}(\eta) = \frac{2}{u+1} + \frac{2}{u^2+u} + \frac{2}{u^3+u} H_n\left(-u^{-1}\right). \tag{2.7}$$

From (2.7), we procure the following corollary.

Corollary 1. *For $n \in \mathbb{N}$, we have*

$$\int_{\mathbb{Z}_p} u^{\eta} (1-\eta)^n \, d\mu_{-1}(\eta) = \frac{2}{u+1} + \frac{2}{u^2+u} + \frac{2}{u^3+u} H_n\left(-u^{-1}\right).$$

3. Some identities on the Frobenius-Euler numbers

In this section, we develop Frobenius-Euler numbers, that is, we derive some interesting and worthwhile relations for studying in Theory of Analytic Numbers.

Now also, for $x \in [0,1]$, we rewrite definition of Bernstein polynomials as follows:

$$B_{k,n}(x) = \binom{n}{k} x^k (1-x)^{n-k}, \text{ where } n,k \in \mathbb{Z}_+. \tag{3.1}$$

By expression of (3.1), we have the properties of symmetry of Bernstein polynomials as follows:

$$B_{k,n}(x) = B_{n-k,n}(1-x), \text{ (for detail, see [21])}. \tag{3.2}$$

Thus, from Corollary 1, (3.1) and (3.2), we see that

$$\begin{aligned}
\int_{\mathbb{Z}_p} B_{k,n}(\eta) u^\eta d\mu_{-1}(\eta) &= \int_{\mathbb{Z}_p} B_{n-k,n}(1-\eta) u^\eta d\mu_{-1}(\eta) \\
&= \binom{n}{k} \sum_{l=0}^{k} \binom{k}{l} (-1)^{k+l} \int_{\mathbb{Z}_p} u^\eta (1-\eta)^{n-l} d\mu_{-1}(\eta) \\
&= \binom{n}{k} \sum_{l=0}^{k} \binom{k}{l} (-1)^{k+l} \left(\frac{2}{u+1} + \frac{2}{u^2+u} + \frac{2}{u^3+u} H_{n-l}(-u^{-1}) \right).
\end{aligned}$$

For n, $k \in \mathbb{Z}_+$ with $n > k$, we compute

$$\begin{aligned}
&\int_{\mathbb{Z}_p} B_{k,n}(\eta) u^\eta d\mu_{-1}(\eta) \\
&= \binom{n}{k} \sum_{l=0}^{k} \binom{k}{l} (-1)^{k+l} \left(\frac{2}{u+1} + \frac{2}{u^2+u} + \frac{2}{u^3+u} H_{n-l}(-u^{-1}) \right) \\
&= \begin{cases} \frac{2}{u+1} + \frac{2}{u^2+u} + \frac{2}{u^3+u} H_n(-u^{-1}), & \text{if } k=0, \\ \binom{n}{k} \sum_{l=0}^{k} \binom{k}{l} (-1)^{k+l} \left(\frac{2}{u+1} + \frac{2}{u^2+u} + \frac{2}{u^3+u} H_{n-l}(-u^{-1}) \right), & \text{if } k>0. \end{cases}
\end{aligned} \tag{3.3}$$

Let us take the fermionic p-adic q-integral on $\mathbb{Z}_p$ on the Bernstein polynomials of degree n as follows:

$$\begin{aligned}
\int_{\mathbb{Z}_p} B_{k,n}(\eta) u^\eta d\mu_{-1}(\eta) &= \binom{n}{k} \int_{\mathbb{Z}_p} \eta^k (1-\eta)^{n-k} u^\eta d\mu_{-1}(\eta) \\
&= \frac{2}{u+1} \binom{n}{k} \sum_{l=0}^{n-k} \binom{n-k}{l} (-1)^l H_{l+k}(-u^{-1}).
\end{aligned} \tag{3.4}$$

Consequently, by expression of (3.3) and (3.4), we state the following Theorem:

Theorem 3. *The following identity holds true:*

$$\sum_{l=0}^{n-k} \binom{n-k}{l} (-1)^l H_{l+k}(-u^{-1}) = \begin{cases} 1 + u^{-1} + u^{-2} H_n(-u^{-1}), & \text{if } k=0, \\ \sum_{l=0}^{k} \binom{k}{l} (-1)^{k+l} \left(1 + u^{-1} + u^{-2} H_{n-l}(-u^{-1})\right), & \text{if } k>0. \end{cases}$$

Let $n_1, n_2, k \in \mathbb{Z}_+$ with $n_1 + n_2 > 2k$. Then, we derive the followings

$$\begin{aligned}
& \int_{\mathbb{Z}_p} B_{k,n_1}(\eta) B_{k,n_2}(\eta) u^\eta d\mu_{-1}(\eta) \\
= & \binom{n_1}{k}\binom{n_2}{k} \sum_{l=0}^{2k} \binom{2k}{l} (-1)^{2k+l} \int_{\mathbb{Z}_p} (1-\eta)^{n_1+n_2-l} u^\eta d\mu_{-1}(\eta) \\
= & \left(\binom{n_1}{k}\binom{n_2}{k} \sum_{l=0}^{2k} \binom{2k}{l} (-1)^{2k+l} \left(\frac{2}{u+1} + \frac{2}{u^2+u} + \frac{2}{u^3+u} H_{n_1+n_2-l}\left(-u^{-1}\right) \right) \right) \\
= & \begin{cases} \frac{2}{u+1} + \frac{2}{u^2+u} + \frac{2}{u^3+u} H_{n_1+n_2}\left(-u^{-1}\right), & \text{if } k=0, \\ \binom{n_1}{k}\binom{n_2}{k} \sum_{l=0}^{2k} \binom{2k}{l} (-1)^{2k+l} \left(\frac{2}{u+1} + \frac{2}{u^2+u} + \frac{2}{u^3+u} H_{n_1+n_2-l}\left(-u^{-1}\right) \right), & \text{if } k \neq 0. \end{cases}
\end{aligned}$$

Therefore, we obtain the following Theorem:

Theorem 4. *For $n_1, n_2, k \in \mathbb{Z}_+$ with $n_1 + n_2 > 2k$, we have*

$$\begin{aligned}
& \int_{\mathbb{Z}_p} B_{k,n_1}(\eta) B_{k,n_2}(\eta) u^\eta d\mu_{-1}(\eta) \\
= & \begin{cases} \frac{2}{u+1} + \frac{2}{u^2+u} + \frac{2}{u^3+u} H_{n_1+n_2}\left(-u^{-1}\right), & \textit{if } k=0, \\ \binom{n_1}{k}\binom{n_2}{k} \sum_{l=0}^{2k} \binom{2k}{l} (-1)^{2k+l} \left(\frac{2}{u+1} + \frac{2}{u^2+u} + \frac{2}{u^3+u} H_{n_1+n_2-l}\left(-u^{-1}\right) \right), & \textit{if } k \neq 0. \end{cases}
\end{aligned}$$

By using the binomial theorem, we can derive the following equation.

$$\begin{aligned}
& \int_{\mathbb{Z}_p} B_{k,n_1}(\eta) B_{k,n_2}(\eta) u^\eta d\mu_{-1}(\eta) \\
= & \prod_{i=1}^{2} \binom{n_i}{k} \sum_{l=0}^{n_1+n_2-2k} \binom{n_1+n_2-2k}{l} (-1)^l \int_{\mathbb{Z}_p} \eta^{2k+l} u^\eta d\mu_{-1}(\eta) \\
= & \frac{2}{u+1} \prod_{i=1}^{2} \binom{n_i}{k} \sum_{l=0}^{n_1+n_2-2k} \binom{n_1+n_2-2k}{l} (-1)^l H_{2k+l}\left(-u^{-1}\right).
\end{aligned} \tag{3.5}$$

Thus, we can obtain the following Corollary:

Corollary 2. *For $n_1, n_2, k \in \mathbb{Z}_+$ with $n_1 + n_2 > 2k$, we have*

$$\begin{aligned}
& \sum_{l=0}^{n_1+n_2-2k} \binom{n_1+n_2-2k}{l} (-1)^l H_{2k+l}\left(-u^{-1}\right) \\
= & \begin{cases} 1 + u^{-1} + u^{-2} H_{n_1+n_2}\left(-u^{-1}\right), & \textit{if } k=0, \\ \sum_{l=0}^{2k} \binom{2k}{l} (-1)^{2k+l} \left(1 + u^{-1} + u^{-2} H_{n_1+n_2-l}\left(-u^{-1}\right)\right), & \textit{if } k \neq 0. \end{cases}
\end{aligned}$$

For $\eta \in \mathbb{Z}_p$ and $s \in \mathbb{N}$ with $s \geq 2$, let $n_1, n_2, ..., n_s, k \in \mathbb{Z}_+$ with $\sum_{l=1}^{s} n_l > sk$. Then we take the fermionic p-adic q-integral on $\mathbb{Z}_p$ for the Bernstein polynomials

of degree n as follows:

$$
\begin{aligned}
& \int_{\mathbb{Z}_p} \underbrace{B_{k,n_1}(\eta) B_{k,n_2}(\eta) \dots B_{k,n_s}(\eta)}_{s-times} u^{\eta} d\mu_{-1}(\eta) \\
= & \prod_{i=1}^{s} \binom{n_i}{k} \int_{\mathbb{Z}_p} \eta^{sk} (1-\eta)^{n_1+n_2+\dots+n_s-sk} u^{\eta} d\mu_{-1}(\eta) \\
= & \prod_{i=1}^{s} \binom{n_i}{k} \sum_{l=0}^{sk} \binom{sk}{l} (-1)^{l+sk} \int_{\mathbb{Z}_p} (1-\xi)^{n_1+n_2+\dots+n_s-l} u^{\eta} d\mu_{-1}(\eta) \\
= & \begin{cases} \frac{2}{u+1} + \frac{2}{u^2+u} + \frac{2}{u^3+u} H_{n_1+n_2+\dots+n_s}\left(-u^{-1}\right), & \text{if } k=0, \\ \prod\limits_{i=1}^{s} \binom{n_i}{k} \sum_{l=0}^{sk} \binom{sk}{l} (-1)^{sk+l} \left(\frac{2}{u+1} + \frac{2}{u^2+u} + \frac{2}{u^3+u} H_{n_1+n_2+\dots+n_s-l}\left(-u^{-1}\right)\right), & \text{if } k \neq 0. \end{cases}
\end{aligned}
$$

So from above, we have the following Theorem:

Theorem 5. *For $s \in \mathbb{N}$ with $s \geq 2$, let $n_1, n_2, \dots, n_s, k \in \mathbb{Z}_+$ with $\sum_{l=1}^{s} n_l > sk$. Then we have*

$$
\begin{aligned}
& \int_{\mathbb{Z}_p} u^{\eta} \prod_{i=1}^{s} B_{k,n_i}(\eta) u^{\eta} d\mu_{-1}(\eta) \\
= & \begin{cases} \frac{2}{u+1} + \frac{2}{u^2+u} + \frac{2}{u^3+u} H_{n_1+n_2+\dots+n_s}\left(-u^{-1}\right), & \text{if } k=0, \\ \prod\limits_{i=1}^{s} \binom{n_i}{k} \sum_{l=0}^{sk} \binom{sk}{l} (-1)^{sk+l} \left(\frac{2}{u+1} + \frac{2}{u^2+u} + \frac{2}{u^3+u} H_{n_1+n_2+\dots+n_s-l}\left(-u^{-1}\right)\right), & \text{if } k \neq 0. \end{cases}
\end{aligned}
$$

From the definition of Bernstein polynomials and the binomial theorem, we easily get

$$
\begin{aligned}
& \int_{\mathbb{Z}_p} \underbrace{B_{k,n_1}(\eta) B_{k,n_2}(\eta) \dots B_{k,n_s}(\eta)}_{s-times} u^{\eta} d\mu_{-1}(\eta) \\
= & \prod_{i=1}^{s} \binom{n_i}{k} \sum_{l=0}^{n_1+\dots+n_s-sk} \binom{\sum_{d=1}^{s}(n_d-k)}{l} (-1)^l \int_{\mathbb{Z}_p} \xi^{sk+l} u^{\eta} d\mu_{-1}(\eta) \\
= & \frac{2}{u+1} \prod_{i=1}^{s} \binom{n_i}{k} \sum_{l=0}^{n_1+\dots+n_s-sk} \binom{\sum_{d=1}^{s}(n_d-k)}{l} (-1)^l H_{sk+l}\left(-u^{-1}\right). \qquad (3.6)
\end{aligned}
$$

Therefore, by (3.6), we get novel properties of Frobenius-Euler numbers with the following corollary:

Corollary 3. *For $s \in \mathbb{N}$ with $s \geq 2$, let $n_1, n_2, \dots, n_s, k \in \mathbb{Z}_+$ with $\sum_{l=1}^{s} n_l > sk$. Then, we have*

$$
\begin{aligned}
u^2 & \sum_{l=0}^{n_1+\dots+n_s-sk} \binom{\sum_{d=1}^{s}(n_d-k)}{l} (-1)^l H_{sk+l}\left(-u^{-1}\right) \\
= & \begin{cases} u^2 + u + H_{n_1+n_2+\dots+n_s}\left(-u^{-1}\right), & \text{if } k=0, \\ \sum_{l=0}^{sk} \binom{sk}{l} (-1)^{sk+l} \left(u^2 + u + H_{n_1+n_2+\dots+n_s-l}\left(-u^{-1}\right)\right), & \text{if } k \neq 0. \end{cases}
\end{aligned}
$$

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University of Gaziantep, Faculty of Science and Arts, Department of Mathematics, 27310 Gaziantep, TURKEY
E-mail address: `mtsrkn@hotmail.com`

University of Gaziantep, Faculty of Science and Arts, Department of Mathematics, 27310 Gaziantep, TURKEY
E-mail address: `acikgoz@gantep.edu.tr`